Problem 4.6

Using Equation 4.32 and footnote 5, show that

$$Y_\ell^{-m} = (-1)^m \left(Y_\ell^m\right)^*.$$

Solution

Equation 4.32 on page 137 is the formula for the spherical harmonics.

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, e^{im\phi} P_\ell^m(\cos\theta) \tag{4.32}$$

Footnote 5 is on page 135, and it gives the associated Legendre functions if m is negative.

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

Therefore,

$$\begin{aligned}
Y_\ell^{-m}(\theta,\phi) &= \sqrt{\frac{(2\ell+1)}{4\pi}\frac{[\ell-(-m)]!}{[\ell+(-m)]!}}\, e^{i(-m)\phi} P_\ell^{-m}(\cos\theta) \\
&= \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}}\, e^{-im\phi} \left[(-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta)\right] \\
&= (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, e^{-im\phi} P_\ell^m(\cos\theta) \\
&= (-1)^m \left[\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, e^{im\phi} P_\ell^m(\cos\theta)\right]^* \\
&= (-1)^m \left[Y_\ell^m(\theta,\phi)\right]^*.
\end{aligned}$$